

The Luther-Emery liquid: Spin gap and anomalous flux period

Alexander Seidel and Dung-Hai Lee

*Department of Physics, University of California at Berkeley, Berkeley, CA 94720, USA and
Materials Sciences Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA*

(Dated: February 2, 2008)

We study the dependence of the ground state energy on an applied Aharonov-Bohm flux Φ for the Luttinger model with large momentum scattering. Employing the method of finite size bosonization, we show that for systems with a spin gap but with gapless charge degrees of freedom, the ground state energy has an exact period of $hc/2e$, i. e. *half* a flux quantum, in the limit of large system size L . Finite size corrections are found to vanish exponentially in L . This behavior is contrasted to that of the spin gapless case, for both even and odd particle number. Generalizations to finite temperature are also discussed.

PACS numbers: 71.10.Pm, 71.10.Hf

I. INTRODUCTION

Models of interacting electrons in one spatial dimension are very valuable for the understanding of strongly correlated systems. This is because there exist theoretical methods enabling us to determine their physical properties reliably. Indeed, by combining perturbative renormalization group¹, bosonization, and Bethe ansatz techniques, a wealth of interesting phases in one dimension has been discovered.

While some properties of these phases are unique in one dimension, others have their higher dimensional analogs. For example the independent gapless spin- and charge-excitations and the vanishing quasiparticle weight of the *Luttinger liquid*² are unique in 1D. However, the fact that it has a finite charge compressibility and Drude weight is analogous to a normal metal in higher dimensions. As another example, like systems in higher dimensions, a Mott insulating state is realized at half filling for repulsive interactions. However, the fact that antiferromagnetic long range order is absent and that spin 1/2 excitation exists in the half-filled Mott state are special features of 1D.

Furthermore, in one dimension there exists a phase, the *Luther-Emery liquid*³, which exhibits a spin gap and no charge gap. In addition, as in the *Luttinger liquid*, the DC electric conductivity is infinite. The above characteristics suggest that the *Luther-Emery liquid* is a 1D analog of a superconductor. However, up until very recently an important question remained unanswered: “Do electrons pair in the *Luther-Emery liquid*?” The best way to answer that question is to determine whether the magnetic flux period is hc/e or $hc/2e$.^{4,5} However, since the spin and charge degrees of freedom are manifestly separate in the effective theory describing the *Luther-Emery liquid*, and the vector potential enters only in the charge action, it is difficult to see why the flux period for a *Luttinger liquid* and a *Luther-Emery liquid* should be different.

In a recent paper we addressed these issues in the one-dimensional t - J - J' -model in the limit of vanishing exchange couplings.⁶ Fortunately, both a spin gapless phase

as well as a spin gapped phase appear in this limit.⁷ In Ref. 6 we have demonstrated that while the flux period is $\Phi_0 \equiv hc/e$ in the former⁸, it indeed becomes $hc/2e$ in the latter. In particular we have shown that as a function of the Aharonov-Bohm flux, the ground state energy of a spin gapped ring is periodic with period $hc/2e$. Due to one-dimensionality the energy barrier between adjacent minima is proportional to the inverse circumference L of the ring. For definiteness, we therefore define the function

$$\mathcal{E}(\Phi) = \lim_{L \rightarrow \infty} L (E_0(\Phi) - E_0(0)) \quad (1)$$

where $E_0(\Phi)$ is the ground state energy of the system as function of flux.

Despite the above progress, the question “do all *Luther-Emery liquids* exhibit an $hc/2e$ flux period, and hence electron pairing?” remains to be answered. In this paper, we show that the answer to the above question is indeed affirmative. Technically we start from the Luttinger Hamiltonian with the g_1 channel scattering.¹ We bosonize this model using the constructive formalism^{9,10} which provides rigorous operator identities on the Hilbert space of the finite size system. We show that due to a set of constraints on the total charge/spin number/current operators^{2,11}, the state of the spin sector impacts the charge sector through a twisting of the boundary condition. As a result, when the spin sector is gapped by the large momentum transfer two body scattering, the charge channel flux period becomes $hc/2e$.

In the literature, the fact that there exist constraints on the total charge/current operators in bosonization has been employed by Loss¹² for spinless fermion systems to study particle number parity effects. Regarding spinful fermions, Ref. 13 used a method similar to ours to determine the flux period for the Hubbard model. However, the author concluded that the flux period is always $hc/2e$ regardless of whether a spin gap exists, which we believe to be in error. Furthermore, a common reasoning encountered in the literature is to attribute the $hc/2e$ flux period to the dominance of singlet

superconducting (SS) correlations at long distance and low energy, rather than to the appearance of a spin gap. It has, however, been noted that states with dominant charge density wave (CDW) correlations may also feature this anomalous flux period (see, e. g., Refs. 6,14,15). Here we argue that this is just the case when there is a spin gap. In this case, it is natural to interpret the state as being formed by Cooper pairs. The degree of coherence of these pairs will determine if the state is more appropriately thought of as CDW-like or SS-like on not too large length scales. In this picture, one naturally expects the flux period to be one half of a flux quantum. In the following, we will show that regardless of the correlation functions in the charge sector, the existence of a spin gap *alone* indeed causes the $hc/2e$ flux period in systems with even particle number.

The structure of this paper is as follows: In section II we present the Luttinger model with large momentum scattering and state the selection rules between charge and current quantum numbers that characterize its Hilbert space. In section III we briefly review the formalism of constructive bosonization and introduce some notation. In section IV we complete the proof that the flux period will be $hc/2e$ in the presence of a spin gap, and contrast this behavior with that expected in the spin gapless case for even and odd particle number. We will also comment on finite temperature effects here. Our conclusions are summarized in section V. Appendix A discusses the finite size refermionization of the spin part of the Hamiltonian, supplementing our line of arguments given in the bulk of this paper. Appendix B is devoted to the use of conjugate phase variables in the construction of Klein factors.

II. THE MODEL AND THE SELECTION RULES

The Tomonaga-Luttinger Hamiltonian describes a gas consisting of right and left moving chiral fermions, each suffering small-momentum transfer scattering in a one-dimensional system of size L :

$$H_{TL} = H_0 + H_2 + H_4 \quad (2)$$

$$H_0 = \sum_{r,k,s} (rv_F k - \mu) : c_{rks}^\dagger c_{rks} : \quad (3)$$

$$H_2 = \frac{1}{L} \sum_{q,s,s'} (g_{2||} \delta_{s,s'} + g_{2\perp} \delta_{s,-s'}) \rho_{+,s}(q) \rho_{-,s'}(-q) \quad (4)$$

$$H_4 = \frac{1}{2L} \sum_{r,q,s,s'} (g_{4||} \delta_{s,s'} + g_{4\perp} \delta_{s,-s'}) : \rho_{r,s}(q) \rho_{r,s'}(-q) : \quad (5)$$

Here $k = 2\pi n/L$ denotes the allowed momenta under periodic boundary condition, the fermion operator c_{rks} annihilates a right ($r = +$) or left ($r = -$) moving fermion with momentum k and spin s (see Fig. 1), $\mu = v_F \pi/L$,

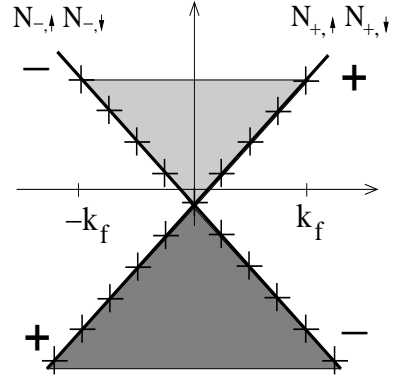


FIG. 1: Right- and left-moving branches of the Luttinger model. The crosses denote the allowed momenta $k = 2\pi n/L$ for periodic boundary conditions. The dark shaded region represents the occupied momentum states in the “vacuum” $N_{r,s} = 0$. The vacuum chemical potential lies between the last occupied and first unoccupied states as indicated by the horizontal line. The light shaded region corresponds to a different filling. The Fermi momentum corresponding to the latter is given by $k_f = 2\pi N_{r,s}/L$.

and

$$: \mathcal{O} : \equiv \mathcal{O} - \langle \mathcal{O} \rangle_0. \quad (6)$$

In the above, $\langle \dots \rangle_0$ denotes the expectation value taken in the vacuum state defined as the ground state of Eq. (3). The density operators appearing in Eqs. (2)-(5) are defined as

$$\rho_{r,s}(q) \equiv \sum_k : c_{r,k+q,s}^\dagger c_{r,k,s} : \quad (7)$$

The $q=0$ -component of these operators,

$$N_{r,s} \equiv \rho_{r,s}(0) \quad (8)$$

measures the extra number of (r,s) type fermions added on top of the vacuum. All four integers $N_{r,s}$ are conserved by H_{TL} . These quantum numbers play an important role in the rest of the paper. Their importance in the bosonization procedure has been stressed by Heidenreich et al.⁹ and Haldane¹⁰.

Out of the four operators $N_{r,s}$ we can form the following linearly independent number and current operators:

$$\begin{aligned} N_\rho &= \sum_{r,s} N_{r,s} \quad , \quad J_\rho = \sum_{r,s} r N_{r,s} \\ N_\sigma &= \sum_{r,s} s N_{r,s} \quad , \quad J_\sigma = \sum_{r,s} rs N_{r,s}, \end{aligned} \quad (9)$$

where the indices ρ and σ stand for charge and spin respectively. It will be important in the following to note that in any one-band model with single particle states symmetrically occupied between $k_f = 2\pi N_{r,s}/L$ and $-k_f$, the total particle number is actually given by

$$N = 2 + \sum_{r,s} N_{r,s} = 2 + N_\rho \quad (10)$$

The reason for this is that the states at $k = 0$, which consist of 4 degenerate states in the Luttinger model rather than 2, have not been included in the definition of the $N_{r,s}$ (see Fig. 1).

There are important relations between the integer quantum numbers defined in Eq. (9). For example $N, N_{\rho,\sigma}, J_{\rho,\sigma}$ are either all odd or all even. In addition, the average of N_ρ and J_ρ has the same even-odd parity as the average of N_σ and J_σ , while they both have opposite even-odd parity as the average of N and J_ρ . These constraints are summarized by the following “selection rules”¹¹:

$$\begin{aligned} (-1)^N &= (-1)^{N_\rho} = (-1)^{J_\rho} = (-1)^{N_\sigma} = (-1)^{J_\sigma} \quad (11a) \\ -(-1)^{(N+J_\rho)/2} &= (-1)^{(N_\rho+J_\rho)/2} = (-1)^{(N_\sigma+J_\sigma)/2} \quad (11b) \end{aligned}$$

which follow from the definitions Eq. (9) and the fact that the $N_{r,s}$ are integer. For most of the paper, we shall primarily concentrate on the case where N is even. While selection rule Eq. (11a) then requires the same of all the other quantum numbers, it is the selection rule Eq. (11b) that imposes a coupling between the spin and charge quantum numbers which ultimately determines the value of the flux period.

The Tomonaga-Luttinger Hamiltonian Eq. (2) is exactly solvable.^{9,10,16} The solution describes a system with gapless spin and charge excitations. A spin gap may be opened by the addition of the following large-momentum transfer scattering term³:

$$\begin{aligned} H &= H_{TL} + H_1 \quad (12) \\ H_1 &= H_{1,||} + H_{1,\perp} \\ &= -\frac{1}{L} \sum_{k,k',q,s,s'} (g_{1||} \delta_{ss'} + g_{1\perp} \delta_{-ss'}) \\ &\quad \times : c_{+,k'+q,s'}^\dagger c_{+,k,s} :: c_{-,k-q,s}^\dagger c_{-,k',s'} : \quad (13) \end{aligned}$$

When the number of particles is incommensurate with the number of lattice sites, Eq. (12) is the generic Hamiltonian including the most relevant two-body scattering terms. The inclusion of H_1 destroys the exact solubility of the model, and at the same time it destroys the conservation of J_σ . However since H_1 changes J_σ in multiples of 4, the parity $(-1)^{J_\sigma/2}$ remains conserved. As a result the selection rules Eq. (11a), Eq. (11b) remain valid even in the presence of H_1 .

III. BOSONIZATION

Under suitable choices of parameters, Eq. (12) can describe a translationally invariant system of spin-1/2 fermions with a spin gap but no charge gap, i.e., a *Luther-Emery liquid*. In the rest of the paper we study the dependence of the ground state energy of such a model as a function of an applied Aharonov-Bohm flux. Technically

we employ the constructive bosonization method^{9,10} extensively reviewed in Refs. 17 and 18. In the following we shall just summarize the main bosonization rules.

Due to the following commutation relation between the density operators

$$[\rho_{r,s}(-rk), \rho_{r',s'}(r'k')] = \frac{kL}{2\pi} \delta_{rr'} \delta_{ss'} \delta_{k,k'}. \quad (14)$$

we define boson creation operators for each momentum $q \neq 0$ and each spin s :

$$\begin{aligned} b_s^\dagger(q) &= \sqrt{\frac{2\pi}{|q|L}} \sum_r \Theta(rq) \rho_{r,s}(q) \\ [b_s(q), b_{s'}^\dagger(q')] &= \delta_{s,s'} \delta_{q,q'}, \quad [b_s(q), b_{s'}(q')] = 0, \quad (15) \end{aligned}$$

where $\Theta(x)$ is the Heaviside step function. The bosonization of the local fermion operators

$$\psi_{r,s}(x) = \frac{1}{\sqrt{L}} \sum_k e^{ikx} c_{r,s}(k) \quad (16)$$

then proceeds by means of the introduction of a non-Hermitian bosonic field

$$\varphi_{r,s}(x) = -\frac{\pi r x}{L} N_{r,s} + i \sum_{q \neq 0} \sqrt{\frac{2\pi}{L|q|}} \Theta(rq) e^{iqx - rq\alpha/2} b_s(q), \quad (17)$$

in terms of which the fermion creation operators can be written^{9,10,17} as:

$$\psi_{r,s}^\dagger(x) = \frac{1}{\sqrt{L}} A_{r,s} e^{i\varphi_{r,s}^\dagger(x)} e^{i\varphi_{r,s}(x)} e^{i\overline{\varphi}_{r,s}}, \quad (18)$$

where the factor

$$A_{r,s} = e^{i\frac{\pi}{2}(r \sum_{s'} N_{-r,s'} + s \sum_{s'} N_{r,s'})}, \quad (19)$$

is introduced to ensure the proper anticommutation relations between the fermion operators (18) carrying different r and s . It commutes with all the spatially dependent fields in Eq. (18). A positive infinitesimal α was introduced in Eq. (17) to ensure the convergence of commutators between operators. The operator $\overline{\varphi}_{r,s}$ is conjugate to $N_{r,s}$,

$$[\overline{\varphi}_{r,s}, N_{r,s}] = i. \quad (20)$$

Note that the validity of Eq. (20) formally requires $N_{r,s}$ to have a continuous spectrum (see Appendix B). This is clearly not the case in the physical Hilbert space \mathcal{H}_{phys} we have been working in so far. We find it convenient, however, to introduce a larger Hilbert space \mathcal{H} , where the $N_{r,s}$ operators have a continuous spectrum. This construction is analogous to the embedding of a discrete lattice into a continuous space, and is reviewed in Appendix B. To ensure that the Hamiltonian, as well as physical observables, do not lead out of \mathcal{H}_{phys} , the operators $\overline{\varphi}_{r,s}$

may only enter through integer powers of the unitary operators $\exp(i\bar{\varphi}_{r,s})$, which raise the $N_{r,s}$ by 1. We shall have occasion though, e. g. in Appendix A, to work in a larger subspace of \mathcal{H} defined below. Formally, it is most convenient to define operators that are valid everywhere in \mathcal{H} . The anticommuting operators $A_{r,s} \exp(i\bar{\varphi}_{r,s})$ are also known as *Klein factors* in the literature.

It is customary to further define local Hermitian fields each associated with the spin (σ) or charge (ρ) degrees of freedom,

$$\begin{aligned}\phi_{\rho,\sigma}(x) &= \frac{1}{4} \sum_r r (\varphi_{r,\uparrow}(x) \pm \varphi_{r,\downarrow}(x) + \text{h.c.}) + \bar{\phi}_{\rho,\sigma} \\ \bar{\phi}_{\rho,\sigma} &= \frac{1}{4} \sum_r r (\bar{\varphi}_{r,\uparrow} \pm \bar{\varphi}_{r,\downarrow}),\end{aligned}\quad (21)$$

as well as their “dual” fields,

$$\begin{aligned}\theta_{\rho,\sigma}(x) &= \frac{1}{4} \sum_r (\varphi_{r,\uparrow}(x) \pm \varphi_{r,\downarrow}(x) + \text{h.c.}) + \bar{\theta}_{\rho,\sigma} \\ \bar{\theta}_{\rho,\sigma} &= \frac{1}{4} \sum_r (\bar{\varphi}_{r,\uparrow} \pm \bar{\varphi}_{r,\downarrow})\end{aligned}\quad (22)$$

Written in terms of the spin and charge boson operators

$$b_{\rho,\sigma}(q) = \frac{1}{\sqrt{2}} (b_{\uparrow}(q) \pm b_{\downarrow}(q)), \quad (23)$$

the above local fields read ($\nu = \rho, \sigma$)

$$\begin{aligned}\phi_{\nu}(x) &= \bar{\phi}_{\nu} - \frac{\pi N_{\nu} x}{2L} - \frac{i}{2} \sum_{q \neq 0} \text{sgn}(q) \sqrt{\frac{\pi}{L|q|}} \\ &\times e^{-iqx - |q|\alpha/2} (b_{\nu}^{\dagger}(q) + b_{\nu}(-q))\end{aligned}\quad (24)$$

$$\begin{aligned}\theta_{\nu}(x) &= \bar{\theta}_{\nu} - \frac{\pi J_{\nu} x}{2L} - \frac{i}{2} \sum_{q \neq 0} \sqrt{\frac{\pi}{L|q|}} \\ &\times e^{-iqx - |q|\alpha/2} (b_{\nu}^{\dagger}(q) - b_{\nu}(-q)).\end{aligned}\quad (25)$$

From Eqs. (24) and (25) it is evident that N_{ν} and J_{ν} are the winding numbers of ϕ_{ν} and θ_{ν} respectively, and $\bar{\phi}_{\nu}$ and $\bar{\theta}_{\nu}$ are the spatial averages of ϕ_{ν} and θ_{ν} . It is simple to check that $\bar{\phi}_{\nu}$ and $\bar{\theta}_{\nu}$ are the conjugate operators of J_{ν} and N_{ν} respectively, i.e.,

$$[\bar{\phi}_{\nu}, J_{\nu}] = i, \quad [\bar{\theta}_{\nu}, N_{\nu}] = i. \quad (26)$$

Note that although the commutation relations (26) are analogous to Eq. (20), the operators $e^{i\bar{\phi}_{\nu}}$ and $e^{i\bar{\theta}_{\nu}}$ lead out of the physical subspace. This is so since within \mathcal{H}_{phys} , the quantum numbers N_{ν} , J_{ν} cannot be raised or lowered by 1 independently, but are subjected to the selection rules Eq. (11). Within this space, only powers of $e^{4i\bar{\phi}_{\nu}}$ and $e^{4i\bar{\theta}_{\nu}}$ are allowed.¹⁹ However, within the larger space \mathcal{H} introduced above, the operators $e^{i\bar{\phi}_{\nu}}$ and $e^{i\bar{\theta}_{\nu}}$ are nonetheless well defined objects. It is convenient to introduce a space of “fractional” excitations, \mathcal{H}_{frac} , generated by acting on \mathcal{H}_{phys} with all possible combinations

of $e^{i\bar{\phi}_{\nu}}$, $e^{i\bar{\theta}_{\nu}}$. Within \mathcal{H}_{frac} , the quantum numbers N_{ν} , J_{ν} are independent integers. We must bear in mind, though, that all physically acceptable states live in \mathcal{H}_{phys} .

The inclusion of the zero modes $\bar{\phi}_{\nu}$ and $\bar{\theta}_{\nu}$ in Eqs. (24) and (25) ensures the proper commutation relations of these fields when the system size L is finite

$$[\phi_{\nu}(x), \theta_{\nu'}(x')] = i \frac{\pi}{4} \delta_{\nu,\nu'} \text{sgn}(x - x'). \quad (27)$$

Eq. (27) suggests that the conjugate operator of $\phi_{\nu}(x)$ is proportional to $\partial_x \theta_{\nu}(x)$, i.e.,

$$\begin{aligned}\Pi_{\nu}(x) &= -\frac{2}{\pi} \partial_x \theta_{\nu}(x) \\ [\phi_{\nu}(x), \Pi_{\nu'}(x')] &= i \delta_{\nu,\nu'} \delta(x - x').\end{aligned}\quad (28)$$

Similarly the conjugate operator of $\theta_{\nu}(x)$ is proportional to $\partial_x \phi_{\nu}(x)$, i.e.,

$$\nu(x) = -\frac{2}{\pi} \partial_x \phi_{\nu}(x) \quad (29)$$

$$[\theta_{\nu}(x), \nu'(x')] = i \delta_{\nu,\nu'} \delta(x - x'). \quad (30)$$

The physical spin or charge density is given by Eq. (29), and in the absence of an applied flux, the physical (spin or charge) current density is given by

$$j_{\nu}(x) = K_{\nu} v_{\nu} \Pi_{\nu}(x) = -\frac{2}{\pi} K_{\nu} v_{\nu} \partial_x \theta_{\nu}(x), \quad (31)$$

which follows from the bosonized Hamiltonian given below. In the above expressions

$$\begin{aligned}v_{\nu} &= \sqrt{\left(v_F + \frac{g_{4\nu}}{\pi}\right)^2 - \left(\frac{g_{2\nu}}{\pi}\right)^2} \\ K_{\nu} &= \sqrt{\frac{\pi v_F - g_{2\nu} + g_{4\nu}}{\pi v_F + g_{2\nu} + g_{4\nu}}} \\ g_{2\rho,\sigma} &= \frac{g_{2||} \pm g_{2\perp} - g_{1||}}{2} \\ g_{4\rho,\sigma} &= \frac{g_{4||} \pm g_{4\perp}}{2}.\end{aligned}\quad (32)$$

Since Π_{ρ} is just the density of right moving fermions minus the density of left moving fermions, it is appropriate to interpret the coefficient $K_{\rho} v_{\rho} \equiv v_F^*$ in Eq. (31) as the renormalized Fermi velocity of the system.

In terms of $\phi_{\nu}(x)$ and $\theta_{\nu}(x)$ the bosonization identity Eq. (18) reads

$$\psi_{r,s}^{\dagger}(x) = \frac{1}{\sqrt{L}} A_{r,s} : e^{i(\theta_{\rho}(x) + r\phi_{\rho}(x) + s(\theta_{\sigma}(x) + r\phi_{\sigma}(x)))} : \quad (33)$$

Here, $: () :$ denotes boson normal ordering: all powers of the fields $\varphi_{r,s}^{\dagger}$ are to be moved to the left of powers of the fields $\varphi_{r,s}$, whereas positive powers of the operator $\exp(i\bar{\varphi}_{r,s})$ are to appear on the very right, and negative powers of the same operator are to appear on the very left of the expression.

By means of Eq. (33) the selection rules Eq. (11) become equivalent to the requirement

$$\psi_{r,s}(x) = \psi_{r,s}(x+L) \quad \forall r, s. \quad (34)$$

This clearly illustrates the topological origin of these rules.

We may now write the Hamiltonian Eq. (12) entirely in terms of the bosonic fields introduced above. The Tomonaga-Luttinger part of the Hamiltonian, including the large momentum scattering term with parallel spin, takes the following quadratic form:

$$\begin{aligned} H_{TL} + H_{1,\parallel} &= \sum_{\nu} v_{\nu} \left\{ \sum_{q \neq 0} |q| \tilde{b}_{\nu}^{\dagger}(q) \tilde{b}_{\nu}(q) + \frac{\pi}{4L} (N_{\nu}^2/K_{\nu} + J_{\nu}^2 K_{\nu}) \right\} \\ &= \sum_{\nu} \frac{v_{\nu}}{\pi} \int dx : \left\{ K_{\nu} (\partial_x \theta_{\nu}(x))^2 + \frac{1}{K_{\nu}} (\partial_x \phi_{\nu}(x))^2 \right\} : \end{aligned} \quad (35)$$

$$(36)$$

where the operators $\tilde{b}_{\nu}(q)$ are related to those in Eq. (23) by a Bogoliubov transformation. The large momentum-transfer scattering with antiparallel spin term becomes³:

$$H_{1,\perp} = -\frac{2g_{1\perp}}{L^2} \int dx : \cos(4\phi_{\sigma}(x)) : \quad (37)$$

Note that the coefficient of 4 in the argument of the cosine assures that the operator does not go out of the physical subspace, as explained above.

The weak coupling renormalization group flow of the system Eq. (12) is well known^{1,20}. For $K_{\sigma} < 1$, the operator $H_{1,\perp}$ is relevant and a spin gap will be opened. This is the case we will focus on in the following. For spin $SU(2)$ invariant systems $g_{i\parallel} = g_{i\perp} \equiv g_i$. In that case $K_{\sigma} < 1$ requires $g_1 < 0$, as discussed by Luther and Emery.³

IV. THE FLUX PERIOD

A. The spin gapped case

By virtue of Eqs. (36) and (37), the model Eq. (12) takes the form

$$H = H_{TL} + H_1 \equiv H_{\rho} + H_{\sigma} \quad (38)$$

where H_{ρ} and H_{σ} act exclusively on charge- and spin-degrees of freedom respectively. The eigenstates are thus direct products of charge states and spin states

$$|c\rangle \otimes |s\rangle, \quad (39)$$

and the ground state energy is the sum of spin and charge energies

$$E_0 = E_0^c + E_0^s. \quad (40)$$

When H_1 causes a spin gap to open up, the spin sector of the model Eq. (12) is described by a gapped sine-Gordon field theory.

In the following we shall focus on the $N_{\sigma} = 0$ sector, which is where the gapped spin ground state lies. In this sector $H_{1,\perp}$ is relevant, and one may interpret the cosine term in Eq. (37) as a steep potential experienced by $\phi_{\sigma}(x)$. In the limit of infinite system size where true symmetry breaking is possible, one may think of $\phi_{\sigma}(x)$ as being locked to one of the minima of the cosine potential. When this happens $\bar{\phi}_{\sigma}$, the spatial average of $\phi_{\sigma}(x)$, will take a c-number value equal to the respective minimum value of ϕ_{σ} . At first, let us neglect the selection rules Eq. (11). That is, we start by looking at the problem in the space \mathcal{H}_{fac} introduced in Section III, where in particular J_{σ} is an independent integer valued quantum number. Then we may regard the conjugate variable $\bar{\phi}_{\sigma}$ as an angular variable with period 2π . This notion becomes precise if we identify $\bar{\phi}_{\sigma}$ with its “lattice version” discussed in Appendix B, which we shall do for the present purpose.²¹ Within $[0, 2\pi)$ there are four inequivalent minima of the cosine term in Eq. (37), and the corresponding ground states in the spin sector can be labeled as

$$|0\rangle, \quad |\pi/2\rangle, \quad |\pi\rangle, \quad |3\pi/2\rangle, \quad (41)$$

where

$$\bar{\phi}_{\sigma} |\phi\rangle = \phi |\phi\rangle. \quad (42)$$

As discussed earlier, the operator

$$\hat{\eta} \equiv (-1)^{J_{\sigma}/2} \equiv \exp(-i\pi J_{\sigma}/2) \quad (43)$$

commutes with H_{σ} , hence its eigenvalues can be used to classify the spin ground states. Unfortunately the states given in Eq. (41) are not eigenstates of $\hat{\eta}$. Following Appendix B, it is easy to show that

$$\bar{\phi}_{\sigma} \hat{\eta} |\phi\rangle = \left(\phi + \frac{\pi}{2}\right) \hat{\eta} |\phi\rangle, \quad (44)$$

where the eigenvalue on the right hand side is to be understood modulo 2π . We may hence choose the global phases in Eq. (41) such that

$$|z\pi/2\rangle = \hat{\eta}^z |0\rangle. \quad (45)$$

It is thus easy to form linear combinations

$$|\eta\rangle = \sum_{z=0}^3 \eta^{-z} \hat{\eta}^z |0\rangle \quad (46)$$

such that

$$\hat{\eta} |\eta\rangle = \eta |\eta\rangle. \quad (47)$$

We are now in a position to enforce the selection rules (11). Given $N_\sigma = 0$, the selection rule Eq. (11a) requires J_σ to be even. As a result only $\eta = \pm 1$ are allowed. We label these two states by

$$|+\rangle, |-\rangle. \quad (48)$$

Thus actually, the ground state is only two-fold degenerate. This degeneracy becomes further lifted in the case of a finite system size L , to be discussed next.

For finite L , the notion that the field ϕ_σ is locked to a classical value is no longer valid. In fact for finite L , even $\bar{\phi}_\sigma$ is subjected to quantum fluctuations. This is explicit in Eq. (35), where the variable conjugate to $\bar{\phi}_\sigma$, namely J_σ , enters the Hamiltonian when L is finite. Thus the spin ground state can no longer be thought of as one of the “locked” spin states given by Eq. (41). On the other hand, since η remains a good quantum number, the states in Eq. (48) are still well defined as the respective ground states in the $\eta = \pm$ sectors of the spin Hilbert space. We note that the spin states in Eq. (48) thus defined are not strictly degenerate for finite L . It is important to observe, however, that the difference in energy between these two states vanishes exponentially in the system size L . One way to see this is the well known fact that the gapped sine-Gordon field theory is the low energy effective theory of a dimerized spin-1/2 chain.²² Here, the $|\pm\rangle$ are respectively the symmetric and anti-symmetric combination of the two dimer patterns. Since the two dimer patterns differ by a macroscopic number of degrees of freedom, the tunnel splitting between these two states should vanish exponentially with the system size. A slightly more direct way to see the above is offered by the well known mapping between the gapped sine-Gordon theory and the massive Thirring model.^{3,23} We will elaborate on this point in Appendix A. The advantage of this method is that at the special Luther-Emery point, it allows us to study the effect of a finite temperature.

For the purpose of this paper we may ignore the above exponentially small energy difference between the states Eq. (48). This is because such a tiny difference will drop out of in the limit taken in Eq. (1). In this sense we may still speak of a degeneracy in the spin sector of the model, and regard the spin contribution E_0^s in Eq. (40) as essentially independent of η in the spin gapped case.

Naively the spin degeneracy discussed above seems to suggest that the ground state of the full Hamiltonian Eq. (38) is degenerate. However this is not so, and the reason for this is the selection rule Eq. (11b). To demonstrate that let us assume the total particle number to be $N = 4m + 2$, whereas $N_\sigma = 0$. According to the selection rule Eq. (11b) the spin states $|\pm\rangle$ may not be combined with the same charge state. The spin state $|-\rangle$ may only be combined with a charge state whose current quantum number J_ρ is an odd multiple of 2 and hence non-zero. The presence of a non-zero current will cost an energy of order v_F^*/L as is evident from Eq. (35). The state $|+\rangle$, on the other hand, may be combined with a charge state

of zero current, which minimizes the charge energy. As a result there is an energy splitting $\sim 1/L$ between the lowest energy state in the $\eta = +$ and $\eta = -$ sectors. We note that an analogous result was discussed by Haldane¹¹ for the case of a vanishing spin gap and a finite charge gap at commensurate band fillings. In contrast, here we are interested in the effect of an applied Aharonov-Bohm (AB) flux, which is of interest only when the charge sector is gapless.

The coupling to a vector potential $A(x)$ is determined by gauge invariance and can be worked out from the minimal coupling requirement. We only consider the constant vector potential $A(x) = \Phi/L$ corresponding to an AB flux. The correct coupling to Φ then follows from the formal replacement⁴

$$\psi_{r,s}^\dagger(x) \longrightarrow e^{-i\frac{2\pi}{L}\frac{\Phi}{\Phi_0}x} \psi_{r,s}^\dagger(x) \quad (49)$$

in the Hamiltonian, where a charge $-e$ is assumed. Here, the boundary conditions of the field $\psi_{r,s}^\dagger(x)$ remain the same, while the right hand side of Eq. (49) will in general satisfy different boundary conditions. By Eq. (33), this is equivalent to the following replacement in the Hamiltonian Eq. (36) and the current Eq. (31):

$$\theta_\rho(x) \longrightarrow \theta_\rho(x) - \frac{2\pi}{L} \frac{\Phi}{\Phi_0} x \quad (50)$$

or, by Eq. (25), simply

$$J_\rho \longrightarrow J_\rho + 4 \frac{\Phi}{\Phi_0}. \quad (51)$$

Note that we did not attempt to introduce the gauge flux prior to bosonization. This is due to the fact that the fermionic field theory (2) suffers from the well known chiral anomaly. The latter renders the global current of the model ambiguous in the presence of a general AB flux, unless gauge invariance is manifestly enforced. Through the “Lenz rule” $I = -c \partial E_0(\Phi)/\partial \Phi$ (Kohn, Ref. 5, see also Eq. (54)), this ambiguity also enters the ground state energy dependence on flux. To deal with this problem will in any case require the use of gauge invariance, and this is most conveniently achieved in the final, bosonic language. It would be interesting to obtain Eqs. (50), (51) via a more “microscopic” route, i. e. via bosonization of a microscopic (lattice) Hamiltonian with flux; this is subject to current investigations. We stress again, however, that Eq. (50) is uniquely determined by the minimal coupling principle.

From Eqs. (35), (51) the energy versus flux function $\mathcal{E}(\Phi)$ in Eq. (1) is given by

$$\mathcal{E}(\Phi) = \min_{J_\rho = \dots -2, 0, 2, \dots} \frac{\pi}{4} v_\rho K_\rho \left(J_\rho + 4 \frac{\Phi}{\Phi_0} \right)^2 \quad (52)$$

Here, all multiples of 2 are allowed values for J_ρ by selection rule Eq. (11a). This leads to the various branches shown in Fig. 2. The alternating labels of $\eta = \pm$ reflect the fact that the spin state has to be adjusted according

to selection rule Eq. (11b) whenever J_ρ is changed by 2. In the presence of a spin gap, however, this does *not* affect the energy in the limit of Eq. (1), as discussed above. As a consequence, Eq. (52) has an exact period equal to half a flux quantum, shown by the lower envelope in Fig. 2a). We note that these findings are in complete agreement with those obtained in Ref. 6 for the t - J - J' model. The amplitude of the ground state energy modulations is apparently given by

$$\Delta E = \frac{\pi}{4L} K_\rho v_\rho = \frac{\pi}{4L} v_F^*. \quad (53)$$

where v_F^* is the renormalized Fermi velocity introduced above. The corresponding modulations of the charge current for a given quantum number J_ρ are given by

$$\begin{aligned} \bar{I}_\rho &= -e \frac{1}{L} \int_0^L dx j_\rho(x) \\ &= \frac{2e}{L\pi} v_F^* \int_0^L dx \left(\partial_x \theta_\nu(x) - \frac{2\pi}{L} \frac{\Phi}{\Phi_0} \right) \\ &= -\frac{ev_F^*}{L} \left(J_\rho + 4 \frac{\Phi}{\Phi_0} \right) \\ &= -c \frac{\partial}{\partial \Phi} \mathcal{E}(\Phi) / L. \end{aligned} \quad (54)$$

The current is thus diamagnetic for $-\Phi_0/2 < \Phi < \Phi_0/2$ and is given by a sawtooth curve in general which one obtains by taking the derivative of the envelope in Fig. 2a). The amplitude of the current is given by $\Delta I = ev_F^*/L$, which is the same as that of spinless particles¹², although the flux period is halved. Note that this observation is consistent with the notion that the charge of the carriers is effectively doubled.

B. The spin gapless case

To establish the fact that the $hc/2e$ flux period is due to the presence of a spin gap, it is prudent to demonstrate the change of flux period when the spin gap collapses. First, let us assume $N = 4m + 2$ as before. Our discussion from the preceding sections generalizes most easily to the case of a vanishing spin gap, if we also assume isotropy in the spin sector: In this case, $SU(2)$ invariance requires the parameter K_σ to be unity at the Luttinger liquid fixed point. We will comment on the general non-isotropic case below.

For gapless spins, the operator $H_{1,\perp}$ in Eq. (38) is irrelevant, and we may expect to get qualitatively correct results by omitting it. With this simplification, the spin sector becomes analogous to the charge sector, and in particular J_σ can be regarded as a good quantum number. The $\eta = +$ spin ground state $|+\rangle$ then has $J_\sigma = 0$, whereas the state $|-\rangle$ lives in a degenerate doublet space with $J_\sigma = \pm 2$. This then raises the corresponding spin energy of the $|-\rangle$ state by a term of order v_σ/L , as shown explicitly in Eq. (35). As a consequence, the $\eta = -$

branches are shifted upward with respect to the $\eta = +$ branches (Fig. 2b)) which destroys the $hc/2e$ periodicity of $\mathcal{E}(\Phi)$.

The spin current carrying $\eta = -$ states may (but need not) be shifted up in energy so much that $\Phi = hc/2e$ ceases to be a metastable minimum of $\mathcal{E}(\Phi)$. This is just the case for a non-interacting system. In the case where the uplifting of the $\eta = -$ state is not as large, $\Phi = hc/2e$ persists to be a metastable minimum in the energy versus flux curve. The difference between the ground state energy at $\Phi = 0$ and $\Phi = hc/2e$ is thus given by:

$$|E_0(\Phi_0/2) - E_0(0)| = \frac{\pi}{L} \min(v_\sigma, K_\rho v_\rho) \quad (\text{isotropic spin}) \quad (55)$$

Interestingly, Eq. (55) provides information about the Luttinger parameters of the *spin sector*. It has long been known that for a Luttinger liquid, the Luttinger parameters can be determined from the ground state properties.² This technique is often applied to infer the charge Luttinger parameters, e. g. by calculating the ground state energy as a function of particle density and magnetic flux.²⁴ Eq. (55) shows that the same technique may be used to infer spin Luttinger parameters, provided that $v_\sigma < K_\rho v_\rho$ holds. In $SU(2)$ invariant systems, the spinon-velocity v_σ may thus be obtained. Note that in this case, the $J_\sigma = \pm 2$, $N_\sigma = 0$ states corresponding to the $\eta = -$ branches in Fig. 2b) are degenerate with states having $J_\sigma = 0$, $N_\sigma = \pm 2$, which carry no spin current but have a net azimuthal spin projection $S_z = \pm 1$. This degeneracy follows from Eq. (35) with $K_\sigma = 1$. The latter states, however, will generally be lower in energy for spin gapless systems *without* $SU(2)$ invariance. This follows because one has $K_\sigma > 1$ in this case, since $K_\sigma < 1$ would always lead to a spin gap. We thus predict that the branches corresponding to the metastable minima in Fig. 2b), if present, will have a net spin, rather than a net spin current, in models without $SU(2)$ invariance. In this case, v_σ in Eq. (55) is to be replaced by v_σ/K_σ .²⁵

When $N = 4m$, all the patterns in Fig. 2 are shifted horizontally by $hc/2e$. In this case the global minima in Fig. 2b) are located at odd multiples of $hc/2e$. Hence the function $\mathcal{E}(\Phi)$ can distinguish the cases $N = 4m$ and $N = 4m + 2$ in the case of gapless spins, but not in the case of gapped spins. The same result had also been observed for the t - J - J' model.⁶

We now turn to the case of odd particle number $N = 2m + 1$. In this case the selection rules Eq. (11a) requires both N_σ and J_σ to be odd, reflecting the fact that there must be a dangling spin. (Of course, with a dangling spin the system cannot have a spin gap.) Now the quantum number $\eta = \exp(-i\pi J_\sigma/2)$ may take the values $\pm i$. The two corresponding subsets of the spin state space are related by the transformation $J_\sigma \rightarrow -J_\sigma$, which leaves the Hamiltonian invariant. As a consequence, the spin ground states $|\eta = \pm i\rangle$ are exactly degenerate, and an exact $hc/2e$ periodicity is obtained for the Hamiltonian (12) at any system size, regardless of whether H_1 is

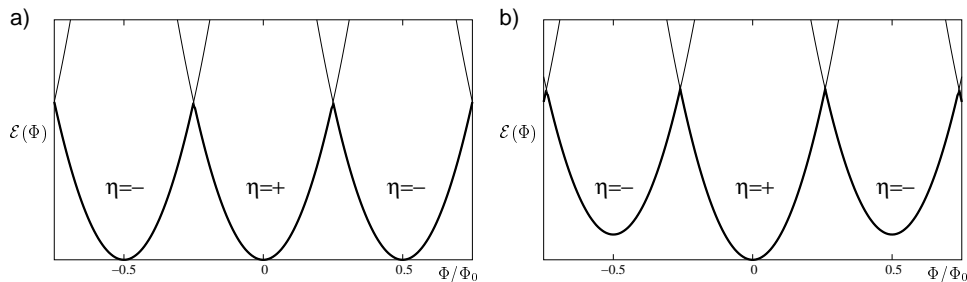


FIG. 2: Energy branches as function of flux for even total particle number N , with and without spin gap. $\mathcal{E}(\Phi)$ is given by the lower envelope. The alternating label $\eta = \pm$ describes the spin state corresponding to each branch, where $N = 4m + 2$ is assumed. a) Spin gapped case. The flux period is $\Phi_0/2$. b) No spin gap. The $\Phi_0/2$ flux periodicity is destroyed by a relative shift between the $\eta = +$ and $\eta = -$ branches.

relevant or not. Also, since J_ρ is now odd as well, the pattern shown in figure 2a) will be shifted horizontally by $hc/4e$. Hence the global minima of $\mathcal{E}(\Phi)$ will be located at odd multiples of $hc/4e$ in this case.

C. Discussion of the results at $T = 0$

The results presented in the preceding section are exact for the Hamiltonian Eq. (12), which is believed to be the low energy effective theory for all one-dimensional systems with gapless, linearly dispersing charge degrees of freedom.²⁶ These results may thus be expected to be representative for this entire universality class. To rigorously justify this point, the effects of higher order, less relevant operators should be included into the model studied above. We will not carry out such a detailed analysis here. Rather, we will point out some expected modifications due to less relevant operators, and argue for the robustness of the basic results derived above by comparing them to special examples of microscopic models, where the features of $\mathcal{E}(\Phi)$ are known analytically or numerically.

The flux period of the repulsive Hubbard model was studied in Ref. 27. These results agree well with our findings for the spin gapless case. In particular, for odd particle number N the global minima of $\mathcal{E}(\Phi)$ are at odd multiples of $hc/4e$, and the flux period is $hc/2e$. While it may seem surprising that the flux period does not distinguish the spin gapless, odd particle number case from the spin gapped case (except for the position of the minima), the microscopic origin of the $hc/2e$ period is of a rather different nature in the two cases. A more subtle effect may demonstrate this: If one calculates the $\mathcal{E}(\Phi)$ of *free* electrons for odd N , one indeed finds that $hc/2e$ is the flux period. However one also finds that there exist corrections to the $hc/2e$ period at order $1/L$ in $\mathcal{E}(\Phi)$. These corrections are due to the band curvature neglected in the Hamiltonian Eq. (12). Similar corrections to $\mathcal{E}(\Phi)$ also exist at odd N for the t - J - J' model. They can be calculated using the method discussed in Ref. 6.²⁸ Such corrections in powers of $1/L$, however, were not found in

the t - J - J' model for the spin gapped case, where only exponentially small corrections were observed. We thus argue that corrections to the $hc/2e$ flux period generally scale as $1/L$ in the odd N case, while they are exponentially small in the spin gapped case. The behavior in the latter case can be attributed to the fact that the spin gap generally causes an exponentially small sensitivity to boundary conditions in the spin sector. This point will be further clarified in Appendix A.

The analysis in the preceding section predicts the ground state to be unique on the branches of $\mathcal{E}(\Phi)$ which contain the global minima. We expect this to be obeyed by general Hamiltonians. However, the four-fold degeneracy which we found between the $N_\sigma = 0$, $J_\sigma = \pm 2$ and $N_\sigma = \pm 2$, $J_\sigma = 0$ states on the metastable branches in the isotropic, even N , gapless spin case is an artifact of our restriction to the Luttinger Hamiltonian Eq. (2). Rather, the true eigenstates are given by a triplet and a singlet to be formed from these four states, giving rise to a small splitting. However, except for this effect, the implied degeneracies at the crossings between the branches remain valid: Although the conservation of J_ρ is approximate once higher order operators are allowed, a change of J_ρ by 2 implies a change of momentum by $2k_f$. Hence the states at a branch crossing will not be mixed, and the cusps in $\mathcal{E}(\Phi)$ will remain sharp for general models.²⁹ Finite size studies of the Hubbard model³⁰ show that the patterns displayed in Fig. 2 indeed emerge very clearly in numerical simulations carried out at moderate system size, both for the spin gapped (attractive) and gapless (repulsive) case.

We thus conclude that all systems which can be characterized as Luther-Emery liquids have the $hc/2e$ flux period. In particular, deviations from the patterns in Fig. 2 such as the appearance of additional minima at higher fractions of a flux quantum must be attributed to finite size effects. Such additional minima at Φ_0/n are known to occur in the large U -limit of the Hubbard model³¹, or the small J -limit of t - J -type models³², for *fixed* system size. The criterion for such finite size effects to disappear is that the amplitude of the oscillations $\Delta E \sim 1/L$ from Eq. (53) is small compared to any other energy scale of

the system. In the above cases, the relevant competing scale is $J \sim t^2/U$. The associated crossover is clearly observed in Ref. 32, where the t - J_z model is studied: For $t/L \lesssim J_z$, the model displays the spin gapped behavior shown in Fig. 2a). This is a consequence of the Ising spin gap of this model. The similar crossover for the repulsive Hubbard model is shown in Ref. 27, where the pattern of Fig. 2b) emerges (with the necessary shift for $N = 4m$). We note that the appearance of local minima separated by half a flux quantum from the global minima in the repulsive Hubbard model is sometimes interpreted as a sign of pairing.³⁰ We stress, however, that this case *does not* meet the criterion of a $\Phi_0/2$ flux period as we define it, since a small splitting of order t^2/U remains between the two types of minima of $\mathcal{E}(\Phi)$, which does not vanish as the system size is taken to infinity.

D. Non-zero temperatures

Finally, we briefly comment on the expected generalization of our findings to finite temperature. The behavior stated below can be verified straightforwardly at the special solvable Luther-Emery point of the model Eq. (12) (see Appendix A). For $T > 0$, we consider the modulations of the free energy $F(T, \Phi)$ as a function of flux. The particle number is held fixed, i.e., the averages are taken in the canonical ensemble. (If the particle number were allowed to fluctuate, the even/odd effects discussed above would considerably weaken the sensitivity to flux.)

Under these conditions, the observations made above for the ground state energy will carry over to the free energy as long as $T < \Delta E = O(1/L)$. However, the limit of Eq. (1) is not to be taken here, because the amplitude of the free energy modulations is proportional to $\exp(-\text{const } T/\Delta E)$ rather than ΔE when $T > \Delta E$. In the spin gapped, even particle number case it remains true that terms violating the $hc/2e$ periodicity are suppressed by a factor $\sim \exp(-\text{const } \Delta_\sigma L/v_\sigma)$, where Δ_σ is the spin gap. Comparing the two exponential factors, these $hc/2e$ violating terms will be negligible until T is of the order of the spin gap. At this temperature, the amplitude of $\mathcal{E}(\Phi)$ is already exponentially suppressed, provided that the system is large enough such that $\Delta E \ll \Delta_\sigma$ is satisfied. This again shows that the $hc/2e$ period will be obeyed as long as ΔE is the smallest energy scale (other than temperature) of the system.

We note that the cusps between the branches in Fig. 2 will be smoothed by thermal fluctuations, giving rise to a finite negative curvature and paramagnetic¹² effects.

V. CONCLUSIONS

In this paper, we demonstrated that the ground state energy of Luther-Emery liquids will generally exhibit an $hc/2e$ flux period. While this statement holds in a strict

sense in the limit of large system size, finite size deviations are expected to be exponentially suppressed in the system size. This result had been anticipated in an earlier work on a particular microscopic realization of the Luther-Emery liquid, the t - J - J' -model.⁶ Here, we generalized the result of Ref. 6 by showing that the $hc/2e$ flux period is implied by the widely accepted low energy effective theory describing such a phase. As a result, we clarify why the state of the spin sector impacts upon the flux period when it is commonly believed that in one dimension spin and charge decouple at low energies. An important aspect of our findings is that in systems with even particle number N , the $hc/2e$ period is triggered by the spin gap (i.e. pairing) alone and is independent of whether the superconducting pair-pair correlations are the dominant long-distance/time correlation function. This may be of particular value for the correct interpretation of numerical work. In addition, we have also discussed the expected finite temperature generalization of our findings.

In Ref. 6 we stressed the $SU(2)$ invariance of the model discussed there. This requirement has been relaxed in the present discussion, where we did not enforce $SU(2)$ invariance. Instead, only the weaker requirement of a conserved z -component of the spin $S_z = N_\sigma/2$ was found necessary. In the anisotropic case, we must also require that the spin gapped ground state has $S_z = 0$ (see footnote 26), which should be automatic in the isotropic case.

Our findings underline the intuitive notion that every spin gapped system with linearly dispersing charge modes should share some features of a superconductor.

Acknowledgments

This work has been supported by DOE grant DE-AC03-76SF00098.

APPENDIX A: REFORMIONIZATION OF THE SPIN HAMILTONIAN FOR FINITE SYSTEM

It is well known that the sine-Gordon model

$$H_\sigma = \frac{v_\sigma}{\pi} \int dx : \left\{ K_\sigma (\partial_x \theta_\sigma(x))^2 + \frac{1}{K_\sigma} (\partial_x \phi_\sigma(x))^2 \right\} : - \frac{2g_{1\perp}}{L^2} \int dx : \cos(4\phi_\sigma(x)) : , \quad (\text{A1})$$

can be mapped onto the massive Thirring model.^{3,23} This mapping permits a rather direct demonstration of the exponentially small energy difference between the $\eta = +$ and $\eta = -$ states discussed in section IV A. In addition, when $K_\sigma = 1/2$, i.e. at the Luther-Emery point, the massive Thirring model reduces to a massive free fermion Hamiltonian which allows the exact calculation of various physical quantities. In particular finite temperature results can be obtained at the Luther-Emery point easily.

In the notation established in section III, the mapping onto the massive Thirring model can be performed by the introduction of the following spinless fermion operators:

$$\tilde{\psi}_r^\dagger(x) = \frac{e^{i\frac{\pi}{L}rx}}{\sqrt{L}} \tilde{A}_r : e^{i(2r\phi_\sigma(x) + \theta_\sigma(x))} : \quad (\text{A2})$$

where $\tilde{A}_r = e^{i\frac{\pi}{2}(\frac{1}{2}rN_\sigma - \frac{1}{4}J_\sigma)}$

Here, the symbol $: :$ denotes a normal ordering convention analogous to that defined in section III, but where $\varphi_{r,s}$ is replaced by the field

$$\begin{aligned} \tilde{\varphi}_r(x) &= \frac{1}{4} \sum_s s \left(3\varphi_{r,s}(x) - \varphi_{-r,s}^\dagger(x) \right) \\ \bar{\tilde{\varphi}}_r &= \frac{1}{4} \sum_s s \left(3\bar{\varphi}_{r,s} - \bar{\varphi}_{-r,s} \right) = 2r\bar{\phi}_\sigma + \bar{\theta}_\sigma \quad . \end{aligned} \quad (\text{A3})$$

It is interesting to note that the operators $\tilde{\psi}_r(x)$, in terms of which the spin Hamiltonian Eq. (A1) is best analyzed, lead out of the physical Hilbert space \mathcal{H}_{phys} : Apart from affecting the spin current quantum number J_σ , they also change the total number of net excited spins, N_σ , by 1. In the physical Hilbert space, such a change must always go along with a change of charge quantum numbers, which are not affected by $\psi_r(x)$. It is quite natural that the action of a single “fractionalized” operator such as $\tilde{\psi}_r(x)$ will lead out of the space of physical states. At this point the larger space \mathcal{H}_{frac} discussed in Section III becomes indispensable, as it allows us to define operators such as $\tilde{\psi}_r(x)$ in the first place. It is clear, however, that these operators enter the Hamiltonian only in appropriate pairs, which leave the physical subspace invariant.

Using standard methods reviewed in Ref. 17, it is straightforward to show that the field defined in Eq. (A2) satisfies the required anticommutation relations. The additional factor $\exp(i\pi rx/L)$ in Eq. (A2) will be commented on below. For now we note that it gives rise to the following boundary conditions for the spinless fermion fields:

$$\tilde{\psi}_r(x+L) = e^{i\pi(N_\sigma + J_\sigma/2 - 1)} \tilde{\psi}_r(x) \quad (\text{A4})$$

As is relevant to section IV, in the following we will concentrate on the case $N_\sigma = 0$ and J_σ even. Eq. (A4) then tells us that the sector $\eta = -$ is represented by fermions obeying periodic boundary conditions, and the $\eta = +$ sector by fermions obeying antiperiodic boundary conditions. Note that without the additional twist in Eq. (A2), it would have been vice versa.

Using the methods discussed in Ref. 17, one may now

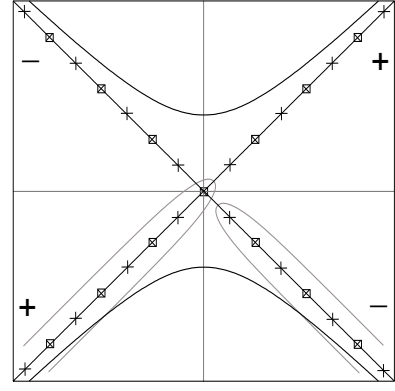


FIG. 3: Ground state of the fermion Hamiltonian Eq. (A5) for $g = 0$. The linear branches indicate the massless case where both N_σ and J_σ are good quantum numbers. Crosses indicate allowed momenta in the $\eta = +$ sector, boxes those in the $\eta = -$ sector. The encircled areas indicate the occupied states on each branch for the ground states in the $N_\sigma = 0$, $J_\sigma = 0$ sector (crosses), as well as the $N_\sigma = 0$, $J_\sigma = +2$ sector (boxes). The latter state is degenerate with the $N_\sigma = 0$, $J_\sigma = -2$ ground state, where the $k = 0$ mode is occupied on the $r = -$ branch instead. For non-zero mass and $N_\sigma = 0$, a gap opens up and the upper band remains empty while the lower band is completely filled, where the allowed momenta depend on η as shown.

show the equivalence of Eq. (A1) and

$$\begin{aligned} H_{mtm} &= \sum_r \int_0^L dx \, (-irv) : \tilde{\psi}_r^\dagger(x) \partial_x \tilde{\psi}_r(x) : \\ &+ \sum_r \int_0^L dx \, m(i)^r : \tilde{\psi}_r^\dagger(x) \tilde{\psi}_{-r}(x) : \\ &+ g \sum_r \int_0^L dx : \tilde{\psi}_r^\dagger(x) \tilde{\psi}_r(x) : : \tilde{\psi}_{-r}^\dagger(x) \tilde{\psi}_{-r}(x) : \\ &\equiv H_\sigma + \text{const}, \end{aligned} \quad (\text{A5})$$

where³³

$$\begin{aligned} v &= \frac{v_\sigma}{4} \left(\frac{1}{K_\sigma} + 4K_\sigma \right) \\ g &= \frac{\pi v_\sigma}{4} \left(\frac{1}{K_\sigma} - 4K_\sigma \right) \\ m &= \frac{|g_{1\perp}|}{2\pi\alpha} \end{aligned} \quad (\text{A6})$$

Here, the symbol $: :$ is as defined in Eq. (6), but the vacuum state is now the $N_\sigma = J_\sigma = 0$ state that is annihilated by the field Eq. (A3).

When $K_\sigma < 1$ the sine-Gordon model is massive, and the physics of Eq. (A5) is given by massive spinless fermions. In particular, the fermionic interaction g vanishes for $K_\sigma = 1/2$, and the Hamiltonian Eq. (A5) becomes that of a massive free fermion model. This is the special point identified by Luther and Emery.³ At

the Luther-Emery point the fermion dispersion relation is given by $\epsilon(k) = \sqrt{(vk)^2 + m^2}$. In a range of K_σ values around $1/2$, the g -term only gives rise to quantitative corrections.

We now return to the factor $\exp(i\pi r x/L)$ in Eq. (A2) (or the boundary conditions Eq. (A4)) and show that with this factor (or the boundary condition specified in Eq. (A4)) the boson and fermion theories are consistent. To illustrate that we compare the ground state degeneracies in the non-interacting massless case for the bosonic and fermionic theories, i. e., we let $K_\sigma = 1/2$ and $g_{1\perp} = 0$, which results in $m = g = 0$.

In the absence of $g_{1\perp}$ both N_σ and J_σ are good quantum numbers. Let us denote the ground state in the N_σ, J_σ sector by $|N_\sigma, J_\sigma\rangle$. From Eq. (35) we recall that $|0, 0\rangle$ is the non-degenerate global ground state of the spin sector, whereas the states $|0, 2\rangle, |0, -2\rangle$ form a degenerate doublet. That this also holds in the fermionic representations of the model is just achieved by the boundary conditions Eq. (A4) (see Fig. 3 and the caption). Note that since the fermions Eq. (A2) are *derived* in terms of bosons, the fermion occupancies claimed below Fig. 3 follow by derivation, not by definition. In fact one may easily evaluate

$$\langle N_\sigma, J_\sigma | \tilde{c}_r^\dagger(k) \tilde{c}_r(k) | N_\sigma, J_\sigma \rangle, \quad (\text{A7})$$

where

$$\begin{aligned} \tilde{c}_r^\dagger(k) &\equiv \frac{1}{\sqrt{L}} \int_0^L dx e^{ikx} \tilde{\psi}_r^\dagger(x) \\ k &= \frac{2\pi}{L} n + \frac{\pi}{L} (N_\sigma + J_\sigma/2 - 1) \end{aligned} \quad (\text{A8})$$

by plugging in Eq. (A2), and verify that the occupancies identified for the various states in Fig. 3 are correct.³⁴

The spin sector of the model is now represented in terms of fermions satisfying a conventional boundary condition given by Eq. (A4). The difference between the ground state energy for $\eta = +$ and $\eta = -$ thus becomes the change in the fermion ground state energy induced by a change of the boundary condition, or equivalently the modulation of the fermion ground state energy caused by an AB flux. When the sine-Gordon model is massive, the fermions form an insulating state. Then, the sensitivity of their ground state energy to the boundary condition will vanish exponentially with the system size, as is well known from the general arguments given by Kohn⁵ and Thouless³⁵. The greatest advantage of the refermionization occurs at the Luther-Emery point. For in that case, a non-trivial interacting bosonic theory is mapped onto a free fermion theory. In particular, at the Luther-Emery point $g = 0$ one obtains from a direct calculation that the energy difference between the ground states for $\eta = +$ and $\eta = -$ vanishes as $m \exp(-mL/v)$, as we claimed earlier.

APPENDIX B: NUMBER AND PHASE VARIABLES FOR CONTINUUM AND LATTICE HILBERT SPACES

The Hilbert space of the Luttinger Hamiltonian Eq. (2), denoted as the “physical” Hilbert space \mathcal{H}_{phys} in the bulk of the paper, can be decomposed as

$$\mathcal{H}_{phys} = \bigotimes_{r,s} \mathcal{H}_l^{N_{r,s}} \otimes \mathcal{H}^{b_{r,s}} \quad (\text{B1})$$

Here, the spaces $\mathcal{H}^{b_{r,s}}$ contain all the degrees of freedom associated with the bosonic excitation spectrum, whereas $\mathcal{H}_l^{N_{r,s}}$ contains the degrees of freedom of the operator $N_{r,s}$. The subscript “ l ” stands for “lattice” and reminds us of the discrete nature of $N_{r,s}$ in the physical Hilbert space: The Hilbert space basis of $\mathcal{H}_l^{N_{r,s}}$ is given by a set of non-degenerate eigenstates of $N_{r,s}$, whose spectrum consists of all integer numbers.

In the process of bosonization, however, we introduce new linear combinations N_ν, J_ν of the $N_{r,s}$ (Eq. (9)). The spectrum of these new operators is likewise integer, yet not all possible combinations of integer eigenvalues are physically allowed. This “residual coupling” is not evident from commutation relations, since the operators N_ν, J_ν all commute as the $N_{r,s}$ do. Hence, once we bosonize an enlarged Hilbert space becomes more natural, where the eigenvalues of the operators N_ν, J_ν are independent. This is the Hilbert space \mathcal{H}_{frac} of “fractional” excitations. The physical subspace \mathcal{H}_{phys} is then characterized by the fact that the selection rules Eq. (11) are satisfied. It is clear that in \mathcal{H}_{frac} , the spectrum of the $N_{r,s}$ must also contain certain fractional values. Formally, we find it convenient to introduce an even larger Hilbert space \mathcal{H} , where the spectrum of the $N_{r,s}$ is *continuous*. The benefit of this is that the conjugate phase $\bar{\varphi}_{r,s}$ of these operators then becomes meaningful. This, in turn, allows us to construct unitary “ladder operators” which change the eigenvalue of the $N_{r,s}$ by *arbitrary* amounts. This formalism is of particular advantage in Appendix A, where fractionalized spin fermion operators are constructed. Below we present some fine details of this embedding of \mathcal{H}_{phys} into the larger space \mathcal{H} .

For this purpose let us consider a single operator \hat{N} and its conjugate variable $\hat{\varphi}$ such that

$$[\hat{\varphi}, \hat{N}] = i \quad (\text{B2})$$

holds. An analogy to the quantum mechanics of a point particle moving in one dimension is obtained if we identify $\hat{N} \equiv \hat{x}$ and $\hat{\varphi} \equiv -\hat{p}$, where \hat{x} and \hat{p} are the coordinate and the momentum of the particle. In this context, it is familiar how to construct a Hilbert space \mathcal{H}^N such that \hat{N} and $\hat{\varphi}$ are well defined on a dense set, Eq. (B2) is satisfied and the spectrum of \hat{N} is unbounded: It is the Hilbert space of square integrable functions of the variable N . From the commutation relation Eq. (B2), it is

clear that the spectrum of both \hat{N} and $\hat{\varphi}$ has to be continuous and unbounded. In particular, we can construct shift operators $\exp(i\hat{\varphi}a)$ satisfying

$$[\hat{N}, e^{i\hat{\varphi}a}] = a e^{i\hat{\varphi}a}, \quad (\text{B3})$$

which shift the value of \hat{N} by an arbitrary amount a .

We note that, as is familiar from the point particle analogy, the “position” and “momentum” eigenkets $|N\rangle$ and $|\varphi\rangle$ are not strictly contained in the Hilbert space \mathcal{H}^N of “proper” vectors, but are “generalized” states in the usual sense. Here, we will not attempt to introduce a different notation for proper and generalized states, nor for the proper Hilbert space and its extension containing generalized states. We refer the reader to Ref. 36 and references therein for details, and simply note that the kets $|N\rangle$ and $|\varphi\rangle$ satisfy

$$\begin{aligned} \langle N' | N \rangle &= \delta(N' - N), \quad \langle \varphi' | \varphi \rangle = 2\pi \delta(\varphi' - \varphi) \\ |\varphi\rangle &= \int dN e^{-i\varphi N} |N\rangle, \quad |N\rangle = \frac{1}{2\pi} \int d\varphi e^{i\varphi N} |\varphi\rangle \end{aligned} \quad (\text{B4})$$

Suppose now that a physical problem is defined on a subspace \mathcal{H}_l^N , which is given by the discrete “lattice” represented by the eigenkets $|N\rangle$ for integer N . Since these kets form a countable Hilbert space basis in \mathcal{H}_l^N , it is natural and convenient to introduce a new scalar product on \mathcal{H}_l^N via:

$${}_l\langle N' | N \rangle_l = \delta_{N', N}. \quad (\text{B5})$$

This differs from the scalar product in \mathcal{H}^N only by an infinite multiplicative factor. In the above, $|N\rangle_l$ denotes the same vector as $|N\rangle$, but endowed with a different scalar product. Eq. (B5) means that the $|N\rangle$ for integer N become a complete orthogonal set of *proper* vectors within \mathcal{H}_l^N . Within \mathcal{H}_l^N , one may now define a “crystal momentum” operator $\hat{\varphi}_l$, whose eigenkets are defined to be

$$|\varphi\rangle_l = \sum_{N \in \mathbb{Z}} e^{-i\varphi N} |N\rangle_l. \quad (\text{B6})$$

One observes that these eigenkets are periodic in φ with period 2π , hence for definiteness the eigenvalues must be restricted to lie within the “Brillouin zone” $(-\pi, \pi]$, where

$${}_l\langle \varphi' | \varphi \rangle_l = 2\pi \delta(\varphi' - \varphi), \quad \varphi', \varphi \in (-\pi, \pi] \quad (\text{B7})$$

holds. If we now denote the restriction of \hat{N} to \mathcal{H}_l^N by \hat{N}_l , we find that the commutator $[\hat{\varphi}_l, \hat{N}_l]$ is *not quite* analogous to Eq. (B2). This has been examined in detail by Schönhammer.^{37,38} However, for the applications we have in mind here, this difference never matters. This is so since all physical observables, including the Hamiltonian, depend on $\hat{\varphi}_l$ only via integer powers of $\exp(i\hat{\varphi}_l)$, and since the equations

$$e^{i\hat{\varphi}} |N\rangle = |N+1\rangle, \quad e^{i\hat{\varphi}_l} |N\rangle_l = |N+1\rangle_l \quad (\text{B8})$$

hold. Hence $\exp(i\hat{\varphi}_l)$ and $\exp(i\hat{\varphi})$ *act identically* on \mathcal{H}_l^N , and we may express all observables in terms of *either* of these operators. By means of Eq. (B6), the kets $|\varphi\rangle_l \in \mathcal{H}_l^N$ are identified with the following kets in \mathcal{H}^N :

$$|\varphi\rangle_l \equiv \sum_{N \in \mathbb{Z}} e^{-i\varphi N} |N\rangle = \sum_{n \in \mathbb{Z}} |\varphi + 2\pi n\rangle \quad (\text{B9})$$

Note that the norm of the right hand side with the scalar product in \mathcal{H}^N formally computes to $\infty \times \delta(0)$, which by comparison with Eq. (B7) is larger by an infinite multiplicative constant than ${}_l\langle \varphi | \varphi \rangle_l$. Recall that the same relation also holds for $\langle N | N \rangle$ and ${}_l\langle N | N \rangle_l$ by definition of the scalar product in \mathcal{H}_l^N . Eq. (B9) makes it clear that the kets $|\varphi\rangle_l \in \mathcal{H}_l^N$ are *not* to be identified with the kets $|\varphi\rangle \in \mathcal{H}^N$. The latter are not periodic in φ , and cannot be constructed solely from kets $|N\rangle$ with integer N . However,

$$\langle \varphi + 2\pi n | N \rangle = {}_l\langle \varphi | N \rangle_l \quad \forall n \in \mathbb{Z} \quad (\text{B10})$$

holds. Thus whenever it is clear from the context that we are working in \mathcal{H}_l^N , we may drop all labels “ l ”, keeping the periodicity of the states $|\varphi\rangle \in \mathcal{H}_l^N$ in mind.

We can now define the Hilbert space \mathcal{H} introduced in the main part of the paper as

$$\mathcal{H} = \bigotimes_{r,s} \mathcal{H}^{N_{r,s}} \otimes \mathcal{H}^{b_{r,s}}. \quad (\text{B11})$$

The advantage of embedding \mathcal{H}_{phys} into the space \mathcal{H} is that now the space \mathcal{H}_{frac} , satisfying $\mathcal{H}_{phys} \subset \mathcal{H}_{frac} \subset \mathcal{H}$, can be generated easily through the action of the operators $\exp(i\bar{\varphi}_\nu)$, $\exp(i\bar{\varphi}_\nu)$ on \mathcal{H}_{phys} . The bookkeeping is greatly simplified by the simple commutation relation of $\bar{\varphi}_{r,s}$ and $N_{r,s}$, Eq. (20), valid on \mathcal{H} . We finally note that \mathcal{H}_{frac} can be written as a product analogous to Eq. (B1), involving a space containing bosonic degrees of freedom and four discrete “lattice” spaces containing the degrees of freedom of the quantum numbers N_ν , J_ν . All the above therefore holds in an analogous way for \mathcal{H}_{frac} , and in particular lattice versions of the phase operators $\bar{\varphi}_\nu$, $\bar{\theta}_\nu$ can be constructed if desired.

In Ref. 38, it has been noted that the use of the canonical commutation relations Eqs. (20), (B2) in constructing the Klein factors does indeed yield correct results in bosonization, even though these relations cannot be rigorously justified for operators that are restricted to a discrete space. We believe that the embedding procedure discussed here provides a proper explanation for this observation.

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